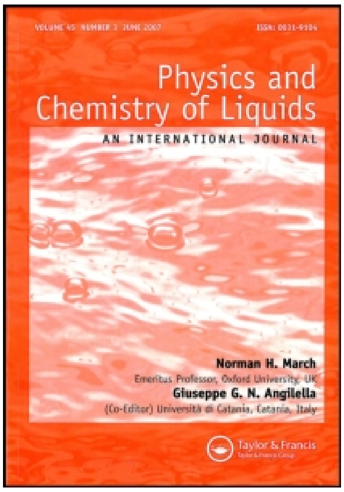

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Review

SUPERFLUIDITY WITH AND WITHOUT A CONDENSATE IN INTERACTING BOSE FLUIDS

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Motivated by recent experiments on alkali gases in atom traps, a largely pedagogical account is given of the implications of the existence of a single-particle Bose–Einstein condensate for the phenomenon of superfluidity. The first conclusion is that, for alkalis in traps at the lowest temperatures, Bose–Einstein condensation coexists with superfluidity. Both experimental evidence and basic microscopic theory are reviewed in this context, with superfluidity in finite systems, quantized vortices and the threshold for breakdown of superfluidity all being referred to. The contrast between liquid ^{4}He below the λ point and the alkali atom gases is then emphasized. Both exhibit superfluidity, but the manifestations of a Bose–Einstein condensate are quite different. For the atomic vapours, near to 100% of the atoms are condensed at the lowest temperatures, whereas from (i) neutron scattering and (ii) computer simulations in liquid ^{4}He present evidence is that 7% is the condensate fraction. This leads finally into a brief discussion of superfluidity without a condensate, with specific reference to the two-dimensional Bose Coulomb gas. The main conclusion is that Bose–Einstein condensation and superfluidity are distinct consequences of deeper topological properties of the many-body wavefunction.

Keywords: Bose–Einstein condensation; Superfluidity; Microscopic theory

PACS Numbers: 05.30.-d, 03.75.Fi, 73.43.Nq, 67.40.Kh

1. INTRODUCTION TO BASIC CONCEPTS OF BOSE–EINSTEIN CONDENSATION AND SUPERFLUIDITY

The phenomenon of Bose–Einstein condensation (BEC) was predicted by Einstein [1,2] in 1924–25 by extending to material particles Bose's work on the statistics of photons: there is no restriction on the occupancy of a single quantum state by bosons. For an ideal gas in a box at temperature $T=0$ in the rest frame the bosons can all condense in the zero-momentum state and the macroscopic occupation of this state starts at a critical temperature T_c given by

$$n\lambda_{dB}^3\Big|_c = 2.612 \tag{1}$$

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for bosons of spin zero. Here n is the particle density and $\lambda_{dB} = \sqrt{(2\pi\hbar^2)/(mk_B T)}$ is the thermal de Broglie wavelength, with m the particle mass, $\hbar$ Planck's constant, and k_B Boltzmann's constant. The relation (1) derives from the vanishing of the chemical potential at T_c and implies that the de Broglie wavelength has become comparable to the mean interparticle separation, so that quantum interference effects between the particles are emerging.

The condensate fraction n_0/n in the ideal Bose gas at $T < T_c$ is

$$n_0/n = 1 - (T/T_c)^{3/2} \tag{2}$$

since $\lambda_{dB} \propto T^{-1/2}$, and tends to unity for $T \to 0$. The concept of Bose–Einstein condensation can be extended to an interacting Bose fluid: the effect of the interactions is to correlate the motions of the particles and hence, by increasing the kinetic energy, to cause a depletion of the condensate even at $T = 0$. The quantum depletion for a weakly-coupled Bose gas with hard-core interactions was estimated by Bogoliubov [3] to be

$$n_0/n|_{T=0} = 1 - (8/3\sqrt{\pi})\sqrt{na^3}. \tag{3}$$

Here a is the diameter of the hard core, and Eq. (3) holds as long as the dilution condition $\sqrt{na^3} \ll 1$ is satisfied.

For a homogeneous interacting Bose gas a general definition of the condensate fraction was given by Penrose and Onsager [4], who associated BEC with the emergence of the *off-diagonal long-range order* in the one-body density matrix $\rho(\mathbf{x}, \mathbf{x}') = \langle \Psi^\dagger(\mathbf{x})\Psi(\mathbf{x}')\rangle$. Here $\Psi^\dagger(\mathrm{x})$ and $\Psi(\mathbf{x})$ are field operators which create and annihilate a particle at $\mathbf{x}$, while$\langle\cdots\rangle$ represents an average on a suitably defined ensemble to take into account the presence of the condensate (see e.g. Hohenberg and Martin [5]). The Fourier transform of the density matrix with respect to the relative distance $\mathbf{x} - \mathbf{x}'$ gives the momentum distribution $n(\mathbf{p})$: therefore the condensate density, being determined by the particles in the state of zero momentum, is given by

$$n_0 = \lim_{|\mathbf{x}-\mathbf{x}'|\to 0} \rho(\mathbf{x}, \mathbf{x}'). \tag{4}$$

The particle density is instead given by

$$n = \lim_{|\mathbf{x}-\mathbf{x}'|\to 0} \rho(\mathbf{x}, \mathbf{x}'). \tag{5}$$

A very clear physical example of BEC is provided by the recent experiments on metastable vapours of alkali atoms inside magnetic or optical traps [6]. As a result of several cooling stages by a combination of laser light and magnetic fields, macroscopic occupation of the lowest level of the harmonic trap was first observed in 1995 as a sudden appearance of a high peak at the center of the density profile of the cloud. Such a dramatic evidence of the condensate is due to the inhomogeneity of the confining potential, which drives the condensate to the center of the trap and allows one to observe "real space" condensation.

The origin of the concept of superfluidity lies in the experiments on liquid ^{4}He [7], which we summarize briefly here. While just below its boiling point ^{4}He behaves as an ordinary fluid with low viscosity, it undergoes a liquid–liquid transition at 2.17 K to a different phase (HeII), the transition being signalled by a specific-heat anomaly whose shape has led to the name λ-line for the coexistence curve of the two phases. The HeII phase was named "superfluid" to describe its peculiar behaviour in transport and excitation experiments, such as non-Newtonian flow ("fountain effect", discovered by Allen and Misener [8]), propagation of heat waves ("second sound", first observed by Peshkov [9]) and sudden arrest of bubbling below the λ-point ("thermal superconductor effect", see for instance Lynton [10]).

Another classical experimental demonstration of superfluidity in liquid HeII is the rotating-bucket experiment [11] first suggested by London (see also the review by Fairbank [12]): on cooling a slowly rotating bucket of liquid ^{4}He below the λ-point, the superfluid portion stops rotating and the bucket rotates faster and faster, since only the normal component is dragged by friction with the rough walls of the container. Similarly, in a classical experiment performed by Andronikashvili [13] in 1946 the effective density of the normal component was measured as a function of temperature from the period of torsional oscillations (and hence the moment of inertia) of a pile of thin metal disks, which were closely spaced to ensure that the normal fluid in the interstices would be dragged along while the superfluid remains stationary. The experiment shows that the superfluid fraction increases from zero at the λ-point to essentially unity near 1K (see Fig. 1).

1.1. Phenomenology: The Two-fluid Model

These effects can be explained by viewing HeII as if it were a mixture of two fluids ("two-fluid model", proposed by Tisza [14] and Landau [15,16]): a normal fluid which possesses Newtonian viscosity, and a superfluid which is capable of frictionless flow through capillaries or past obstacles. A connection between superfluid behaviour in HeII and BEC was historically proposed by London, by noticing the analogy in shape between the specific-heat curves of ^{4}He and of an ideal Bose gas, and that the condition (1) for BEC would yield the correct order of magnitude for the critical temperature heralding the superfluid phase.

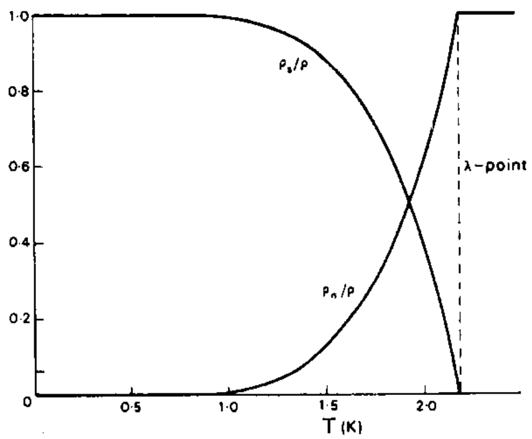

FIGURE 1 Fractions of normal fluid and superfluid in liquid ^{4}He as functions of temperature, from Andronikashvili's experiment.

2. A MICROSCOPIC VIEW OF SUPERFLUIDITY

Given the assumption that the system has a condensate, i.e. that a single-particle state with wavefunction $\Psi_0(\mathbf{r}, t)$ is macroscopically occupied, the conceptual basis of superfluidity can be simply understood in a dilute system (see e.g. Leggett [17,18]). We write $\Psi_0(\mathbf{r}, t) = \sqrt{n_0(\mathbf{r}, t)}\exp[i\phi(\mathbf{r}, t)]$, where n_0 and ϕ are the local density and phase of the condensate, and define the superfluid velocity $\mathbf{v}_s(\mathbf{r}, t)$ by the prescription

$$\mathbf{v}_s(\mathbf{r}, t) = (\hbar/m)\nabla\phi(\mathbf{r}, t). \tag{6}$$

One immediate consequence of the definition (6) is that the superfluid flow is irrotational. In addition, since there is no ''ignorance'' associated with the single quantum state Ψ_0, the entropy has to be carried by the normal component, namely by particles occupying states other than Ψ_0.

Furthermore, from the fact that Ψ_0 must be single-valued *modulo* 2π, one obtains the Onsager–Feynman quantization condition [19,20] on superfluid circulation around a closed circuit:

$$\oint \mathbf{v}_s \cdot d\,\mathbf{l} = qh/m, \tag{7}$$

where q is an integer number. If the superfluid sample is simply connected this condition gives zero, while if vortices are present the superfluid is multiply connected and Eq. (7) is a non-trivial statement: quantization of vortex lines is a peculiarity of superfluidity and exhibits the intrinsically quantum nature of this phenomenon. Experiments on HeII have shown that its vortex lines are indeed quantized [21].

2.1. Microscopic Definitions of Superfluid Density [22]

A first microscopic definition of the superfluid density n_s can be obtained employing a relation due to Josephson and Bogoliubov, which establishes the exact static long-wavelength behaviour of the single-particle Green's function $G_{11}(\mathbf{k}, \omega)$:

$$G_{11}(\mathbf{k}, 0) \rightarrow_{\mathbf{k}\rightarrow 0} n_0 m/(n_s\hbar^2 k^2). \tag{8}$$

At long wavelengths the Green's function is dominated by phase fluctuations: as we shall see below these play a crucial role in reduced dimensionality. In three spatial dimensions the phase–phase correlation function at equal times decays slowly at large distances, thus ensuring the long-range off-diagonal order in Eq. (4).

A second microscopic definition of superfluid density (which may not necessarily coincide with the previous one) comes from the theory of linear response. The current response function, which gives the current density driven by an external field in the linear regime, can be split into its longitudinal and transverse components ($\chi_{\mathrm{L}}(\mathbf{k}, \omega)$ and $\chi_{\mathrm{T}}(\mathbf{k}, \omega)$, say). The density of the normal component of the fluid is given by

$$n - n_s = \lim_{\mathbf{k}\rightarrow 0}\int \frac{d\omega}{2\pi}\frac{\mathrm{Im}\chi_T(\mathbf{k}, \omega)}{\omega}. \tag{9}$$

This sum rule complements the usual f-sum rule on particle conservation, which states that

$$n = \int \frac{d\omega}{2\pi} \frac{\mathrm{Im}\chi_L(\mathbf{k}, \omega)}{\omega}. \tag{10}$$

The fact that the normal-fluid component alone fulfills the sum rule (9) is a microscopic expression of the irrotationality of the superfluid component, as it is probed in a rotating-bucket experiment. It is important to remark that Eq. (9) does not depend explicitly on the condensate density. In Section 5 we will give examples of systems which exhibit superfluid behaviour even in the absence of Bose–Einstein condensation.

It is also possible to provide a third definition of superfluidity by imposing suitable boundary conditions on the many-body wave function (see e.g. Leggett [17]).

2.2. Landau Criterion for Breakdown of Superfluidity

Superfluid flow is experimentally observed in ^{4}He only if the flow rate is lower than a critical velocity v_c at which viscous resistance suddenly appears [7]. The Landau criterion for breakdown of superfluidity [16] states that viscosity starts during flow when the creation of elementary excitations becomes energetically favourable. Let $\varepsilon_{\mathbf{p}}$ be the energy of an elementary excitation with momentum $\mathbf{p}$ in the fluid at rest: if the liquid is dragged with velocity $\mathbf{v}$ then Galileian invariance implies that the excitation energy becomes $\varepsilon_{\mathbf{p}} + \mathbf{p} \cdot \mathbf{v}$, which is negative when $\mathbf{p}$ and $\mathbf{v}$ are antiparallel and the velocity exceeds the critical value $v_c = \varepsilon_{\mathbf{p}}/p$. This implies spontaneous emission of excitations during flow. If the elementary excitations are only of sound-wave type v_c coincides with the velocity of sound; in ^{4}He however the excitation spectrum shows also a "roton" minimum, which accounts for the fact that the critical velocity becomes substantially lower than the velocity of sound.

3. SUPERFLUIDITY IN ATOMIC BOSE–EINSTEIN CONDENSATES

Bose–Einstein condensates of alkali atoms are ultra-cold ($T \simeq 0.1\,\mu\mathrm{K}$), rarefied ($n \simeq 10^{13}\,\mathrm{cm}^{-3}$) clouds of gas contained in magnetic or optical traps. At low temperatures ($T \ll T_c$) there is no discernible non-condensate component and practically all atoms occupy the same quantum level: this implies macroscopic quantum coherence as was verified in interference and Raman- and Bragg-scattering experiments.

3.1. *s*-Wave Scattering Length and Phonon Excitations

Interatomic interactions at such low temperatures and densities are dominated by binary collisions in the s-wave channel, and the interparticle potential is well approximated by a contact interaction determined only by the s-wave scattering length a_s (in the following we shall consider the more common case $a_s > 0$, i.e. repulsive interactions). The atomic cloud is so rarefied that the diluteness condition $\sqrt{na_s^3} \ll 1$ is well satisfied, implying that the mean interparticle distance is much larger than a_s. Under these conditions the system at low temperatures is well described by the Bogoliubov theory for the weakly coupled Bose gas, which implies that the superfluid density coincides with the condensate density. The spectrum of elementary excitations of the gas in the Bogoliubov theory is phonon-like at long wavelength, with a velocity of sound given by $c_B = \sqrt{4\pi\hbar^2 a_s n/m}$, where m is the atomic mass. Thermal excitations with increasing temperature generate a thermal cloud. A picture of a confined condensate and its thermal cloud, as in Fig. 2, thus provides a good visualization of the components of the two-fluid model.

3.2. Superfluidity in Finite Systems [23]

Usually condensates are produced in magnetic traps whose bottom is well approximated by a harmonic potential. The effect of the inhomogeneity of the gas due to the confinement must be taken into account in the theoretical description, leading to quantization of the Bogoliubov sound waves and to new manifestations of superfluidity such as the *scissors modes*, which are also known in nuclear matter.

Scissors modes in atomic gases are excited by a sudden rotation of the anisotropic confining potential [24]. This perturbation gives rise to oscillations at a single frequency in the case of a condensed (superfluid) cloud, but showing beats between two frequencies for a non-condensed (normal-fluid) cloud. While only longitudinal modes can be excited in a superfluid, both longitudinal and transverse modes are excited by the applied perturbation in the normal fluid. This is, in essence, the equivalent of the rotating bucket experiment for a confined superfluid.

3.3. Quantized Vortices

Vortices have been created in atomic condensates with different techniques: by stirring the condensate with laser light or by exploiting interconversion between two components of the condensate with different spins. Figure 3 reproduces three arrays of vortices generated inside confined condensates in the work of the ENS group in Paris [25]: a triangular array (middle image) is the lowest-energy configuration, but a square pattern is also sometimes observed (right image).

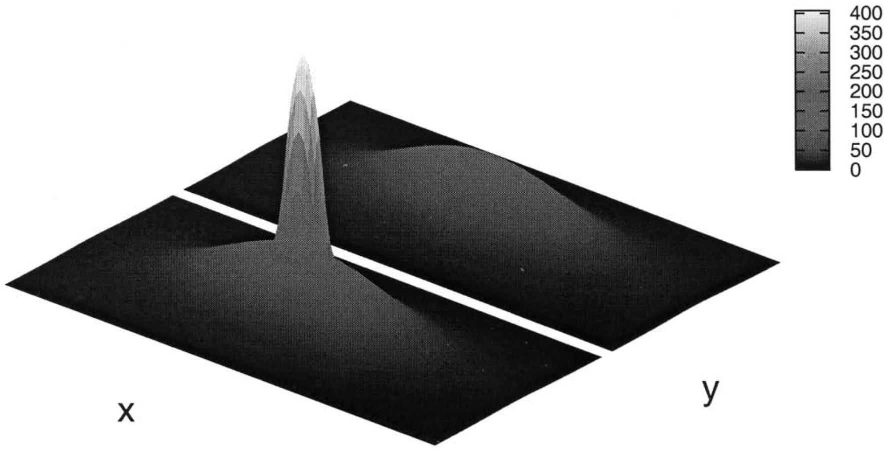

FIGURE 2 Column density profiles for a Bose gas above (back image) and below (front image) the critical temperature for Bose–Einstein condensation.

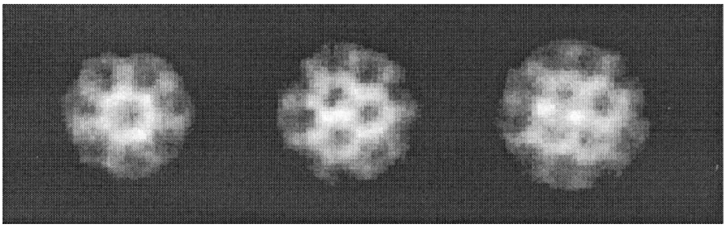

FIGURE 3 Ordered arrays of vortices in a confined Bose–Einstein condensate of ^{87}Rb atoms (from Chevy *et al.* [25]).

Demonstrations that vortices are quantized have also been given. A method used to this end is based on the idea that the presence of a single vortex line inside the condensate breaks time-inversion symmetry and leads in particular to a splitting of the quadrupolar surface modes of azimuthal quantum numbers $m = \pm 2$. As a consequence, a slow precession is induced in a quadrupolar distortion of a condensate containing a vortex at its centre. This can be used to reveal the formation of vortices in an unambiguous manner and also to infer the quantum circulation of the vortex [26]. As an alternative, an interferometric technique has been used to map the phase profile of the condensate wavefunction in a path around the vortex [27].

3.4. Threshold for Breakdown of Superfluidity

The threshold for the breakdown of superfluidity has been investigated in various experiments, probing the condensate both on a macroscopic and on a microscopic scale.

Onofrio *et al.* [28] studied the hydrodynamic flow in a condensate stirred by a blue-detuned laser beam acting as a macroscopic moving object. A density-dependent critical velocity for the onset of a distortion in the density distribution was observed, the distortion being associated with a pressure gradient arising from a drag force between the beam and the condensate. The critical velocity observed was considerably smaller than the local sound velocity, and it was demonstrated that it arises from the periodic shedding of vortex lines at a rate that increases with velocity.

Microscopic probes colliding with a condensate were obtained by adding moving impurities to it [29]. A dramatic reduction of the collision rate was measured when the velocity of the impurity was reduced below the sound velocity, thus providing an experimental test of the Landau criterion for superfluidity.

Breakdown of superfluidity and onset of decoherence have also been observed in a condensate placed inside a magnetic trap with a superimposed optical lattice [30]. The dynamical evolution of the condensate was controlled by displacing the magnetic trap by a variable amount Δx in the direction parallel to the lattice. Different dynamical regimes were observed depending on the magnitude of Δx: while for small displacements the condensate manifests its superfluid behaviour by performing undamped oscillations in the harmonic well, at larger values of the displacement the onset of dissipative processes is observed. The experiments in the dissipative regime can be interpreted in terms of a density-dependent local critical velocity for the destruction of superfluidity given by the local speed of sound, $v_c(x) \propto \sqrt{n(x)}$, thus obtaining a fully quantitative and parameter-free account of data as is reported in Fig. 4.

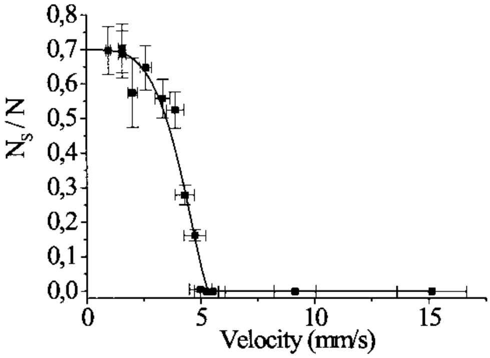

FIGURE 4 The fraction N_s/N of atoms remaining in the undistorted part of a Bose–Einstein condensate of ^{87}Rb atoms driven by a harmonic force through a linear optical lattice, as a function of a maximum velocity reached during motion in the periodic potential (from Burger *et al.* [30]).

4. BOSE–EINSTEIN CONDENSATE IN LIQUID HELIUM: HeII BELOW THE λ POINT

As we have mentioned in Section 1, in liquid ^{4}He superfluidity can be readily demonstrated. However, because this is a strongly-interacting liquid, the condensate fraction is found to be small, less than 10%.

In a homogeneous fluid the condensate fraction is extracted from the distribution $n(\mathbf{p})$ of single-particle momenta as it contributes with a peak at zero momentum,

$$n(\mathbf{p}) = n_0\delta(\mathbf{p}) + \tilde{n}(\mathbf{p}) \tag{11}$$

where $\tilde{n}(\mathbf{p}) = \int d\mathbf{r}[\rho(\mathbf{r}) - n_0]\exp(i\mathbf{p}\cdot\mathbf{r})$ is the momentum distribution of the non-condensate and $\rho(\mathbf{r})$ is the one-body density matrix as a function of the relative coordinate $\mathbf{r}$. Equation (11) immediately yields Eq. (4), since $n(\mathbf{p})$ is the Fourier transform of $\rho(\mathbf{r})$. According to Gavoret and Nozières [31], the condensate fraction also determines a diverging behaviour of $\tilde{n}(\mathbf{p})$ at low momenta,

$$\lim_{p\to 0} p\tilde{n}(\mathbf{p}) = n_0 mc/(2\hbar) \tag{12}$$

where c is the sound velocity of the fluid.

In the experiments the condensate fraction is obtained by neutron inelastic scattering at high energy and momentum transfers [32]. The extraction of $n(\mathbf{p})$ from the measured scattered intensity depends on the experimental resolution and final-state interactions which must be deconvoluted in data analysis. The most recent results on the condensate fraction at $T=0$ obtained by this method yield $n_0/n = (7.25 \pm 0.75)\%$.

Numerical simulations have also been used to predict the condensate fraction and the superfluid density [33]. At $T=0$ variational methods, Green's function Monte Carlo and Diffusion Monte Carlo (DMC) methods have been used and are in reasonable agreement with each other. At finite temperature the Path-Integral Monte Carlo method has been widely employed to characterize the properties of superfluid ^{4}He.

The most recent estimates for the condensate fraction at $T = 0$ by Moroni *et al.* [34] using DMC yield $n_o/n = 7.26\%$, which is in good agreement with the measured value.

5. SUPERFLUIDITY WITHOUT BOSE–EINSTEIN CONDENSATION: THE TWO-DIMENSIONAL BOSE FLUID

Phase fluctuations are enhanced in low-dimensional Bose gases and in particular it is easy to show that the ideal Bose gas in dimensionality D = 2 (2D) does not condense at any finite temperature. We shall consider three different models for the interactions in a 2D Bose gas: (i) a neutral Bose gas with hard-core interactions; (ii) a charged Bose gas with e^2/r interactions; and (iii) a charged Bose gas with $\ln(r)$ interactions. The latter is a model for vortex fluctuations in superfluid or superconducting films [35]. For the case of charged bosons we assume as is customary that they are immersed in a uniform neutralizing background.

Absence of Bose–Einstein condensation in a 2D interacting Bose gas at finite temperature was rigorously proven by Hohenberg using a theorem due to Bogoliubov. He showed that the assumption of long-range off-diagonal order, i.e. $n_0 \neq 0$, would lead to an inconsistency through the following inequality for the momentum distribution at low momenta:

$$\tilde{n}(\mathbf{p}) \geq -\frac{1}{2} + \frac{mk_BT}{p^2}\frac{n_0}{n}. \tag{13}$$

By requiring that the integral of $n(\mathbf{p})$ should be finite (fixed by the particle density) one sees that the only possibility to satisfy the above inequality is to assume that $n_0 = 0$. This conclusion holds for the neutral and the charged Bose gases.

At zero temperature the picture depends on the type of interactions: for the case of the charged Bose gas with ln(r) interactions Magro and Ceperley [36] have used a similar inequality,

$$\tilde{n}(\mathbf{p}) \geq -\frac{1}{2} + \frac{1}{S(\mathbf{p})}\frac{n_0}{n}, \tag{14}$$

to rule out the presence of a Bose–Einstein condensate. This follows directly by noticing that at low momenta the plasma excitations determine the behaviour of the static structure factor $S(p)$ as $S(p) \propto p^2/\Omega$, where $\Omega = \sqrt{2\pi ne^2/m}$ is the plasma frequency. In this system, therefore, the condensate is destroyed by the plasma density fluctuations even at $T = 0$.

The same argument however does not hold for the charged Bose gas with e^2/r interactions, where $S(p) \propto p^{3/2}$, nor for the neutral Bose gas with hard-core interactions, where $S(p) \propto p$. In both cases Eq. (14) gives no inconsistency, and indeed a condensate is found in these fluids at $T = 0$, as in the case of the ideal Bose gas.

Although a condensate is absent in the 2D interacting Bose fluid at finite temperature, the system shows *algebraic* off-diagonal long-range order, i.e. the one-body density matrix decays at large distances through a power law,

$$\rho(\mathbf{r}) \sim \mathrm{r}^{-\alpha}. \tag{15}$$

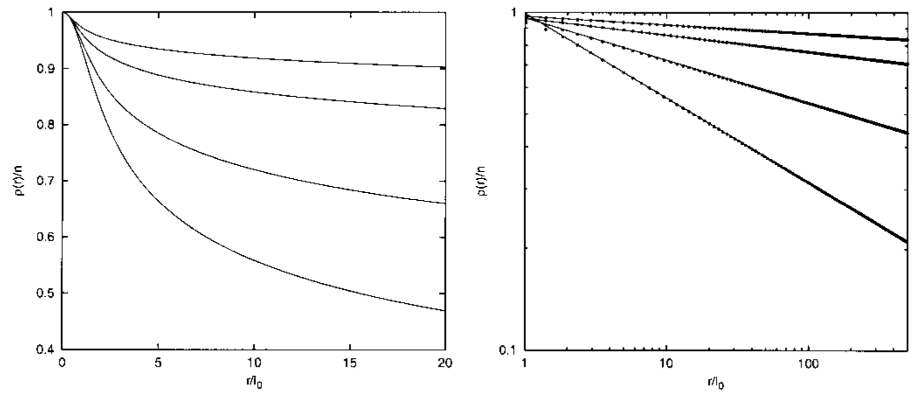

FIGURE 5 The one-body density matrix $\rho(r)/n$ as a function of distance r for a 2D Bose–Coulomb fluid with interaction potential $V(r) = e^2 \ln(r/l_0)$ at zero temperature. Left: for coupling strength $r_s \equiv (2/\pi n l_0^2)^{1/2} = 0.1, 0.2, 0.5$ and 1 (from top to bottom). Right: the same numerical results are plotted in a log–log scale (dots) and compared with analytic results from the correlation function of phase fluctuations (straight lines).

This behaviour is determined by the phase–phase correlations, which decay logarithmically at large distances [37]. The resulting power-law decay of the density matrix is illustrated for the ln(r) fluid in Fig. 5. In the other interacting 2D fluids it is useful to invoke the concept of a quasicondensate, i.e. a condensate with a fluctuating phase [37,38] .Of course, a real finite 2D system might show condensation if its size is smaller than the phase correlation length.

It should be emphasized that interacting 2D fluids do show superfluidity: the transition is predicted to be of the Kosterlitz–Thouless type [39] and in a dilute system at low temperatures the superfluid density can be estimated using the Landau formula,

$$\frac{n_s}{n} = 1 - \frac{1}{nk_BT} \sum_{\mathbf{k} \neq 0} \frac{k^2}{2m} \frac{\exp(\hbar\omega_k/k_BT)}{[\exp(\hbar\omega_k/k_BT) - 1]^2}. \tag{16}$$

This assumes damping of superfluid flow by emission of collective excitations with dispersion relation $\omega_{\mathbf{k}}$ [16].

6. SUMMARY AND FUTURE DIRECTIONS

We have considered in this article only bosonic single-particle condensates, but the discussion could be extended to other related items such as molecular condensates, condensation of excitons in semiconductors or of Cooper pairs in superconductors, or the cross-over from the Bardeen–Cooper–Schrieffer theory of conventional superconductivity to the Bose–Einstein condensation of point-like particles. There also currently is a remarkable effort devoted to the experimental and theoretical study of atomic Fermi gases and boson-fermion mixtures, aimed at realizing novel superfluids from fermion–fermion pairing.

There is now incontrovertible evidence, some of which has been reviewed in Section 3, that Bose–Einstein condensation and superfluidity coexist in gases of alkali atoms in traps. While this evidence comes dominantly from a variety of experiments, all the major predictions of basic microscopic theory are upheld. A topic of great interest at the time of writing concerns the transition from a superfluid to a dissipative regime, either from a dynamical driving of a condensate through an optical lattice as reviewed in Section 3.4 [30] or through the emergence of localization when the potential barriers in an optical lattice or the strength of the repulsive interactions between the atoms are increased [40]. The latter transition has been likened to the quantum phase transition that occurs in a fluid of carriers from a coherent transport regime to a correlation-induced Mott insulator state as may be described by means of a Bose–Hubbard model [41]. Superfluidity in disordered media also presents great fundamental interest. Most appealing are the current endeavours in the development of atom optics, which are aimed towards the realization of "atom lasers" using matter waves.

When one turns to the area more briefly reviewed in Section 4, concerning liquid ^{4}He below the λ line which was historically where the phenomenon of superfluidity was discovered, the theory is less well developed at such a basic microscopic level. The pioneering theoretical studies of Penrose and Onsager in 1956 already contained a "crude estimate" that 8% of the atoms are "condensed", and investigations in the late 1990's now arrive at 7%, both from neutron scattering (Glyde *et al.* [32]) and from Diffusion Monte Carlo calculations (Moroni *et al.* [34]).

This prompts us to comment here, in regard to future directions for further studies on superfluidity in dense quantum liquids, that it is hard to see how such a tiny condensate fraction could be responsible for the dramatic experimental manifestations of superfluidity summarized in Section 1 above. Following Leggett we conclude that it might be that both Bose–Einstein condensation and superfluidity are consequences of deeper topological properties of the many-body wavefunction. As far as these may transpire from numerical studies of the superfluid density and of the one-body density matrix [33], superfluidity is associated with the growth of many-boson exchange processes as temperature decreases, whereas increasing the particle density in a neutral Bose fluid may depress the asymptotic value of the density matrix and hence the condensate fraction.

But be that as it may, it seems of interest here to draw attention again to the potential importance of the chemistry of small Helium clusters, and in particular of $^4He_2$ and $^4He_3$ for constructing such a (say variational) many-body ground-state wavefunction of liquid ^{4}He. Such a discussion began in the 1980s [42–44] and its interest has been re-opened by quite recent studies of helium dimers and trimers in free space. Thus Grisenti *et al.* [45] have reported in the vapour, using a diffraction transmission grating technique, the existence of a Helium dimer with a "bond length" of $\simeq 50$ Å, and as a consequence a very tiny binding energy. The He trimer in free space has also been identified experimentally. Of course, no such free-space information can be quite decisive in regard to the dense liquid. However, experience in nuclear physics with the "alpha particle" model points to the potential usefulness of building especially ^{4}He trimers into a ground-state many-body wavefunction for liquid HeII.

We have two final points to add, again for the future:

(i) As discussed by March and by Ghassib and Chester in the works quoted above, it is possible in such a treatment of the ground state of liquid ^{4}He that so-called Efimov states [46] will play a role. These are states of a three-boson problem when the dimer is either very weakly bound or just unbound. It is relevant in this context, though the system is different, to refer to the investigation of Bulgac [47] on a quantum liquid droplet. He concludes, under some conditions for which the interested reader can refer to the original article, that the effect of three-body correlations can be subsumed into the ground-state energy by using the Efimov states.

(ii) We gave an example in Section 5 of the two-dimensional ln(r) Bose gas in which superfluidity can occur without Bose–Einstein condensation. This strongly motivates, we feel sure, further work on low-dimensional quantum fluids, and in particular the interest in re-opening the early work of Lieb and Liniger [48], with attractive interactions leading to two- and three-body bound states. This may very well prove a link between these points (i) and (ii), with which we conclude this review.

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